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Non-commutative Euclidean structures in compact spaces

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Abstract

Based on results for real deformation parameter q we introduce a compact non-commutative structure covariant under the quantum group $SO_q(3)$ for q being a root of unity. To match the algebra of the q-deformed operators with necessary conjugation properties it is helpful to define a module over the algebra generated by the powers of q. In a representation where X^2 is diagonal we show how P^2 can be calculated. To manifest some typical properties, an example of a one-dimensional q-deformed Heisenberg algebra is also considered and compared with the non-compact case.

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1. Introduction

In papers [1–3] it was shown how the q-deformation of the well known group SO(3) to quantum group $SO_q(3)$ can be used to define a non-commutative quantum space as a comodule of the quantum group. It is very natural to exploit the R matrix as the main tool. Its decomposition into projectors generates a non-commutative (three-dimensional) Euclidean space of coordinates.

Though there have appeared numerous papers dealing with $SO_q(3)$ and $SO_q(N)$ with q being a root of unity the non-commutative structure has been defined for real q only. The value of q becomes important when we demand hermiticity for coordinates (and later on for momenta). For general complex q the R matrix loses its hermiticity, which requires a new definition of conjugation for the coordinate operators. On the other hand there are at least two reasons why one should investigate the case of complex q. First, real q implies always a non-compact coordinate space, while for a compact space we have to admit complex values of q. In context with the fact that non-commutative geometry [4] is considered to be the result of some deep dynamical principle which may be found, e.g. in string theory, the case of compactified dimensions is of special interest. We start here the consideration of an example with only compactified coordinates. The more interesting case with compact and non-compact dimensions (which seems to require different q) is due to further work. Second, we know the quantum group $SO_q(3)$ for generic q and especially the case q being a root of unity, where it demonstrates some peculiarities [5–7]. It is therefore interesting how a non-commutative quantum space can be constructed in that special case. This will be the main aim of our paper.

As we have already mentioned, the key point is the definition of a conjugation for coordinates and momenta, which are later required to be self-adjoint with respect to that conjugation. Different conjugations result in different spaces and hence different physics. The conjugation we will propose below is of course equivalent to ordinary conjugation for real q. We know two ways which are both consistent with $SO_q(3)$. The choice that fits best with our problem is the one where q is left untouched during conjugation. Thus if $\bar{X}$ is the conjugate of an operator X, the conjugate of qX is $q\bar{X}$. This choice has already been used before, for example in [6]. To do this in a mathematically correct way we define a right module over the algebra generated by all powers of q with the additional condition for some power to equal -1 (see next section). The other way, which one may find, i.e. in [7], seems to work better if one deals with non-Hermitian operators having only real eigenvalues, which will not be the case here.

At first our definition looks rather unnatural but in section 2 we shall describe how it works and discuss the consequences. The most important of these is that self-adjoint operators will have (instead of real ones) eigenvalues which are real functions of the parameter q, but this is just what we need, because the scaling operator and its commutation properties force coordinates and momenta to have eigenvalues proportional to powers of q.

The paper is organized as follows. In section 2 we recall the basic formulae for the quantum space of $SO_q(3)$ and state the modifications for our q. In section 3 we consider a one-dimensional example of a q-deformed Heisenberg algebra and demonstrate how it works for q being a root of unity. It is rather helpful to compare our results with earlier ones for real q with the same example. In our main section 4 the non-commutative space covariant under $SO_q(3)$ is considered and matrix elements of coordinates and momenta are calculated. The results are presented explicitly and do not contain any divergencies, which usually occur if one simply replaces q in formulae derived earlier for real q only.

2. Euclidean phase space for q being a root of unity

First we have to recall some basic formulae of the non-commutative space from paper [3], which do not depend on the nature of q. Exploiting the fundamental paper [8] the R matrix of $SO_q(3)$ can be decomposed as

$$\hat{R} = P_5 - \frac{1}{q^4} P_3 + \frac{1}{q^6} P_1. \tag{2.1}$$

We shall not give the projectors P_i here, because we only need P_3. The non-commutative Euclidean space is defined by

$$P_3 XX = 0. \tag{2.2}$$

In the common basis (2.2) looks like

$$\begin{aligned} X^3 X^+ &= q^2 X^+ X^3 \\ X^3 X^- &= q^{-2} X^- X^3 \\ X^- X^+ &= X^+ X^- + \lambda X^3 X^3 \end{aligned} \tag{2.3}$$

where $\lambda = q - \frac{1}{q}$. It is natural to define a metric g_{AB} and an invariant product $X \circ Y$

$$\begin{aligned} & X \circ Y = g_{AB} X^A Y^B \\ & g_{+-} = -q \qquad g_{-+} = -1/q \qquad g_{33} = 1 \end{aligned} \tag{2.4}$$

which let $X \circ X$ commute with X^A. P_3 can be expressed through a generalized ϵ-tensor

$$P_3{}^{AB}{}_{CD} = \frac{1}{1+q^4}\epsilon^{FAB}\epsilon_{FDC} \tag{2.5}$$

where its indices are moved according to formulae such as

$$\epsilon_{ABC} = g_{CD}\,\epsilon_{AB}{}^{D} \tag{2.6}$$

$$\begin{aligned} &\epsilon_{+-}{}^{3} = q \qquad \epsilon_{-+}{}^{3} = -q \qquad \epsilon_{33}{}^{3} = 1 - q^2 \\ &\epsilon_{+3}{}^{+} = 1 \qquad \epsilon_{3+}{}^{+} = -q^2 \\ &\epsilon_{-3}{}^{-} = -q^2 \quad \epsilon_{3-}{}^{-} = 1. \end{aligned} \tag{2.7}$$

Equation (2.3) is then equivalent to

$$X^C X^B \epsilon_{BC}{}^{A} = 0 \tag{2.8}$$

and the R matrix can be expressed in the form

$$\hat{R}^{AB}_{CD} = \delta^A_C\delta^B_D - q^{-4}\epsilon^{FAB}\epsilon_{FDC} - q^{-4}(q^2-1)g^{AB}g_{CD}. \tag{2.9}$$

Now we come to the definition of conjugation. We still choose

$$\overline{X^A} = g_{AB}X^B \equiv X_A \tag{2.10}$$

as in paper [3]. But for generic complex q this is consistent with equations (2.3) only if we define $\bar{q} = q$, which means q is unchanged under conjugation. This forces us to distinguish between q (and its functions) and constant complex numbers which are to be conjugated as usual. (We mean, e.g., the i in the Heisenberg relation, see below.)

That is done best if the vector space the q-deformed operators act on is considered as a (right) module over an algebra A. This associative (and commutative) algebra A over the complex numbers is generated by the powers of q: $q, q^2, \ldots, q^{r-1}$ and the condition $q^r = -1$. The integer r is taken to be larger than two. Within A we define an involution *, which fulfils the usual conditions

$$\begin{aligned} &a^{**} = a \qquad (ab)^* = b^*a^* \\ &(\alpha a + \beta b)^* = \overline{\alpha}a^* + \overline{\beta}b^* \end{aligned} \tag{2.11}$$

where $\alpha, \beta \in C$. These properties are consistent with the choice $q^* = q$, which determines the involution for all elements.

As a next step we consider a right module M over the algebra A. (Since A is commutative an equivalent approach is given considering a left module.) M is a complex vector space. For any $a, b \in A$ and $\eta, \xi \in M$ we have

$$\begin{aligned} &\eta(ab) = (\eta a)b \\ &\eta(a+b) = \eta a + \eta b \\ &(\eta + \xi)a = \eta a + \xi a \end{aligned} \tag{2.12}$$

and any combination of type ηa is again an element of M. For further application we need a Hermitian structure which is created by a Hermitian inner product. For any pair of elements a bilinear map $\langle\eta|\xi\rangle \in A$ is defined with the properties

$$\begin{aligned} &\langle\eta|\xi\rangle^* = \langle\xi|\eta\rangle \\ &\langle\eta a|\xi b\rangle = a^*\langle\eta|\xi\rangle b. \end{aligned} \tag{2.13}$$

A third property, usually required for a Hermitian product, includes the absence of zero-norm states. We shall see below that such states cannot be excluded for our choice of q. Therefore, strictly speaking, our structure is not Hermitian in the usual sense. Nevertheless we keep this terminology but remember that all unusual properties are connected with the existence of zero-norm states. The product allows the definition of the Hermitian conjugation $\overline{O}$ of an operator O

$$\langle\eta|O\xi\rangle = \langle\overline{O}\eta|\xi\rangle. \tag{2.14}$$

We use another symbol so as not to confuse this with the involution in A. Subsequently Hermitian and unitary (isometric) operators are defined. Operators in M can be viewed as matrices with entrances from A: Hermitian conjugation is then transposition together with the involution in A defined above. It is then clear that if λ is an eigenvalue of O then λ^* is an eigenvalue of $\overline{O}$ and hence $\lambda^* = \lambda$ for all eigenvalues of an operator with $\overline{O} = O$. We shall see below that for our q and the operators we are considering it is not necessary to distinguish between Hermitian and self-adjoint operators. Their eigenvalues are real functions of q. One can show directly that the eigenvectors are orthogonal (under the product defined above) for different eigenvalues with the usual arguments. If a unitary operator has an eigenstate ξ with eigenvalue λ one can easily show

$$\langle\xi|\xi\rangle = \lambda^*\lambda\langle\xi|\xi\rangle \tag{2.15}$$

which gives information about λ only for states with non-vanishing norm. This fact becomes important below.

Based on equation (2.10) we can now proceed as in [3] and define a derivative, momentum, angular momentum and the scaling operator Λ in the same way. For the components of the momentum we have the analogue of (2.8), while for angular momentum

$$L^C L^B \epsilon_{BC}{}^A = -1/q^2 W L^A \tag{2.16}$$

and

$$\begin{aligned} &q^4(q^2-1)^2 L\circ L = W^2 - 1 \\ &L^A W = W L^A. \end{aligned} \tag{2.17}$$

The scaling operator Λ is introduced in the same way with the properties

$$\begin{aligned} \Lambda^{1/2}X^A &= q^2 X^A \Lambda^{1/2} \\ \Lambda^{1/2}P^A &= q^{-2} P^A \Lambda^{1/2} \\ \Lambda^{1/2}L^A &= L^A \Lambda^{1/2} \\ \Lambda^{1/2}W &= W\Lambda^{1/2}. \end{aligned} \tag{2.18}$$

Conjugation of vector values is analogous to equation (2.10), W is self-adjoint and Λ is unitary up to normalization:

$$\overline{\Lambda^{1/2}} = q^{-6}\Lambda^{-1/2}. \tag{2.19}$$

Equations (2.16) lead to the standard $\mathrm{SO}_q(3)$ algebra. The generalized Heisenberg relations are

$$P^A X^B - \hat{R}^{-1AB}{}_{CD} X^C P^D = -\frac{\mathrm{i}}{2}\Lambda^{-1/2}\{(1+q^{-6})g^{AB}W - (1-q^{-4})\epsilon^{ABC}L_C\}. \tag{2.20}$$

Now we have to study representations of this algebra. For q being a root of unity the physical relevant representations become finite dimensional while for real q they have infinite dimension. Thus there is no difference here between self-adjoint, essentially self-adjoint and Hermitian operators.

The representations will be studied in detail in section 4.

3. Representations of a one-dimensional q-deformed Heisenberg algebra

We consider now a one-dimensional example of a q-deformed Heisenberg algebra, that is neither a projection of the Euclidean space nor based on the deformation of any symmetry group. It is not even non-commutative in the sense of space coordinates because there is only one. Nevertheless it is based on a modified Leibniz rule and has been studied for real q in great detail [9, 10]. It reflects very nicely the important role which is played by the scaling operator Λ that one has to introduce in a general non-commutative structure of coordinates and momenta. The algebra looks as follows:

$$\begin{aligned}&\frac{1}{\sqrt{q}}PX-\sqrt{q}XP=-\mathrm{i}U\\&UP=qPU\\&UX=\frac{1}{q}XU.\end{aligned}\tag{3.1}$$

Conjugation is given by

$$\begin{aligned}&\bar{P}=P\\&\bar{X}=X\\&\bar{U}=U^{-1}.\end{aligned}\tag{3.2}$$

While there is obviously no problem for real q, with our definition of conjugation for operators and involution of algebra elements equation (3.2) is also consistent with (3.1). To give meaning to operators in our module space we have to enlarge our algebra A to include real functions of q in a straightforward way. We shall consider a representation of the algebra (3.1) based on eigenvectors of P. From the second equation it follows that applying U to such an eigenstate we obtain another one with eigenvalue multiplied by q^{-1}. Therefore we have

$$P|n\rangle^{\pi_0}=\pi_0q^n|n\rangle^{\pi_0}\tag{3.3}$$

where n is an integer, $0\leqslant n\leqslant 2r-1$, and π_0 is an arbitrary real function of q. Further

$$U|n\rangle^{\pi_0}=|n-1\rangle^{\pi_0}\tag{3.4}$$

and according to what was stated in the last section

$$^{\pi_0}\langle n|m\rangle^{\pi_0}=\delta_{nm}.\tag{3.5}$$

Now we have an example where the self-adjoint operator P has eigenvalues that are real functions of q. The powers occurring are a consequence of the properties of U. For our chosen q we can see that the eigenstate $U|0\rangle^{\pi_0}$ has the same eigenvalue as $|2r-1\rangle^{\pi_0}$. Disregarding the case of degeneration we have

$$U|0\rangle^{\pi_0}=C(\pi_0)|2r-1\rangle^{\pi_0}\tag{3.6}$$

where C is a phase factor and different C label different representations. From equations (3.4) and (3.6) we have $U^{2r}=C$ for any state in our representation. Now it is straightforward to define another unitary operator U' by

$$U'=U\mathrm{e}^{-\frac{\mathrm{i}\alpha}{2r}}\tag{3.7}$$

where we have put $C=\mathrm{e}^{\mathrm{i}\alpha}$ Then $U'^{2r}=1$ and it is more convenient to work with a new system $|n\rangle'$

$$U'|n\rangle'=|n-1\rangle'.\tag{3.8}$$

The new eigenstates are just multiplied by phase factors. For shortness we have omitted the upper index π_0. From the first equation of (3.1) and its conjugate one can deduce

$$XP = \frac{\mathrm{i}}{\lambda}\left(\sqrt{q}U - \frac{1}{\sqrt{q}}U^{-1}\right) \tag{3.9}$$

which shows the action of X on the $|n\rangle'$ states:

$$X|n\rangle' = \frac{\mathrm{i}}{q^n\lambda\pi_0}\left(\sqrt{q}\mathrm{e}^{\frac{\mathrm{i}\alpha}{2r}}|n-1\rangle' - \frac{1}{\sqrt{q}}\mathrm{e}^{-\frac{\mathrm{i}\alpha}{2r}}|n+1\rangle'\right). \tag{3.10}$$

This system of $2r$ equations can be solved in principle and the eigenvalues and eigenstates of X can be found, but it is easier to exploit the eigenstates of U, as we shall demonstrate below. We start with

$$\begin{aligned} |\phi_0\rangle &= \sum_{n=0}^{2r-1}|n\rangle' \\ |\phi_k\rangle &= (\pi_0)^{-k}P^k|\phi_0\rangle = \sum_{n=0}^{2r-1}q^{kn}|n\rangle' \end{aligned} \tag{3.11}$$

and integer $0 \leqslant k \leqslant 2r-1$. Obviously

$$U'|\phi_k\rangle = q^k|\phi_k\rangle. \tag{3.12}$$

We mention that for real q those states are non-normalizable, which is not the case here.

Before constructing the eigenstates of X we briefly comment on the eigenstates of U' and U. Our definition of an adjoint operator in section 2 and the inner product lead to unitary operators with respect to that product which will have properties differing from the usual ones, as we have already seen for self-adjoint operators. The eigenstates of our unitary operators may not be orthogonal and can contain zero-norm states. So explicitly

$$\langle\phi_k|\phi_m\rangle = \sum_{n=0}^{2r-1}q^{n(k+m)} \tag{3.13}$$

which is non-zero for $m = k = 0$ or $m+k = 2r$. Hence we have only two non-zero-norm states for $k = 0$ and r and the eigenstates $|\phi_k\rangle$ and $|\phi_{2r-k}\rangle$ for $k = 1, \ldots, r-1$ are not orthogonal.

Keeping all this in mind, we can still work with the states (3.11) as a basis to construct the X eigenstates.

From the algebra (3.1) follows

$$X|\phi_k\rangle = d_k|\phi_{k-1}\rangle \tag{3.14}$$

for $1 \leqslant k \leqslant 2r-1$ and

$$X|\phi_0\rangle = d_0|\phi_{2r-1}\rangle. \tag{3.15}$$

Next we have to calculate d_k. We apply the conjugate of equation (3.9) to $|\phi_k\rangle$ and find

$$d_k = \frac{\mathrm{i}}{\lambda\pi_0}(\mathrm{e}^{\frac{\mathrm{i}\alpha}{2r}}q^{k-\frac{1}{2}} - \mathrm{e}^{-\frac{\mathrm{i}\alpha}{2r}}q^{-k+\frac{1}{2}}). \tag{3.16}$$

This formula works for all $0 \leqslant k \leqslant 2r-1$. We construct the eigenstates in the following way:

$$\begin{aligned} X|x_m\rangle &= x_m|x_m\rangle \\ |x_m\rangle &= \sum_{k=0}^{2r-1}a_k|\phi_k\rangle \end{aligned} \tag{3.17}$$

yielding the recursion relation for the coefficients

$$a_{k+1} = \frac{x_m}{d_{k+1}}a_k. \tag{3.18}$$

Consistency requires

$$a_0 = \frac{x_m}{d_0} a_{2r-1}. \tag{3.19}$$

We can put $a_0 = 1$ and the solution of equations (3.18) and (3.19) are

$$\begin{aligned} a_k &= (x_m)^k \left(\prod_{l=1}^{k} d_l \right)^{-1} \\ (x_m)^{2r} &= \prod_{l=1}^{2r} d_l = \frac{\mathrm{i}^{2r}}{\lambda^{2r} {\pi_0}^{2r}} (-1)^r f^2(q, \alpha) \end{aligned} \tag{3.20}$$

where we have introduced the function

$$f(q, \alpha) = \prod_{k=1}^{r} (q^{k+\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}\alpha}{2r}} - q^{-k-\frac{1}{2}} \mathrm{e}^{-\frac{\mathrm{i}\alpha}{2r}}). \tag{3.21}$$

Equation (3.20) gives (in principle) the possibility of finding the eigenvalues of X. They depend on π_0 and the real function $f^2(q, \alpha)$. The fact that only $(x_m)^{2r}$ is given reflects the property that due to the unitary equivalence of X and P, x_m must be proportional to q^m. Thus equation (3.20) contains no new information we did not have before. The function $f(q, \alpha)$ also occurs in the more realistic three-dimensional case (see next section).

Now we can compare our results with those for real q obtained in papers [9] and [10]. The main difference is that all our representations have finite dimensions, which avoids the mathematical problems of the real case. On the other hand we have to introduce an additional parameter C (or α) characterizing the representation. The operators X and P are manifestly equivalent in our representation.

4. $\mathrm{SO}_q(3)$ deformation in compact space

In this section we give the representations of the q-deformed algebra (2.8), (2.16)–(2.20) for $q^r = -1$. We have not written the $L^A X^B$ and $L^A P^B$ relations, which are the same as in [3]. Nor are we going to repeat the derivations of papers [3] and [11] leading to the T-operators and explaining the appearance of the Clebsch–Gordon coefficients [5, 12] because on the algebraic level there are no changes. The changes start as soon as representations are considered, which shall be done now.

We choose $L \circ L$, L^3 and $X \circ X$ as a complete set of commuting variables. One can proceed as in the undeformed case and exploit equations (2.16) and (2.17). For the angular momentum the eigenvalues are

$$\begin{aligned} L \circ L |j, m, n\rangle &= \frac{q^{-6}}{(q^2 - q^{-2})^2} (q^{4j+2} + q^{-4j-2} - q^2 - q^{-2}) |j, m, n\rangle \\ L^3 |j, m, n\rangle &= -\frac{q^{-3}}{(q - q^{-1})} \left(q^{2m} - \frac{q^{2j+1} + q^{-2j-1}}{q + q^{-1}} \right) |j, m, n\rangle \end{aligned} \tag{4.1}$$

where j and m are integers, $|m| \leqslant j$ and $0 \leqslant j \leqslant j_{\max}$. (Note that the sign of L^3 is opposite to the usual one, because we have kept the conventions of paper [3].) For q being a root of unity we must remember that there are two types of representation, called types I and II in paper [6]. We allow only type II representations for the construction of the non-commutative space. That the type I representations can be omitted consistently follows from paper [7]. The type II representations behave as for $q = 1$ (and general real q) except the fact that

$j_{\max} \leqslant \frac{r}{2} - 1$. The states are fully determined by the quantum numbers j, m and n. From the first equation of (2.13) we read off

$$X^2|j, m, n\rangle = l_0^2 q^{4n}|j, m, n\rangle. \tag{4.2}$$

It is sufficient to choose the integer n as $0 \leqslant n \leqslant r - 1$. The parameter l_0 plays the same role as π_0 in the one-dimensional case.

All our representations are unitary and either irreducible or fully reducible [6]. Irreducible representations are labelled by the integer j. Because of equation (4.2) we deal with finite-dimensional irreducible representations as in the one-dimensional case before. That and the existence of a $j_{\max}$ are the main differences with respect to real q.

The states are normalized in the usual way. The phase factors can be chosen to fulfil

$$\begin{aligned} \Lambda^{\frac{1}{2}}|j, m, n\rangle &= q^{-3}|j, m, n-1\rangle \\ \Lambda^{-\frac{1}{2}}|j, m, n\rangle &= q^{3}|j, m, n+1\rangle. \end{aligned} \tag{4.3}$$

From equation (2.16) the matrix elements of $L^{\pm}$ can be obtained. We mention for further use

$$W|j, m, n\rangle = \frac{\{2j+1\}}{\{1\}}|j, m, n\rangle \tag{4.4}$$

where we have introduced the abbreviations

$$\begin{aligned} \{a\} &= q^a + q^{-a} \\ [a] &= \frac{q^a - q^{-a}}{\lambda}. \end{aligned} \tag{4.5}$$

In papers [3] and [11] one finds how the $\mathrm{SO}_q(3)$ structure can be used to define reduced matrix elements for X^A and P^A. For the non-vanishing matrix elements we quote the results

$$\begin{aligned} \langle j+1, m+1, n|X^+|j, m, n\rangle &= q^{m-2j}\sqrt{[j+m+1][j+m+2]}\langle j+1, n\|X^-\|j, n\rangle \\ \langle j-1, m+1, n|X^+|j, m, n\rangle &= q^{m+2j+2}\sqrt{[j-m][j-m-1]}\langle j-1, n\|X^-\|j, n\rangle \\ \langle j+1, m-1, n|X^-|j, m, n\rangle &= q^{m}\sqrt{[j-m+1][j-m+2]}\langle j+1, n\|X^-\|j, n\rangle \\ \langle j-1, m-1, n|X^-|j, m, n\rangle &= q^{m}\sqrt{[j+m][j+m-1]}\langle j-1, n\|X^-\|j, n\rangle \\ \langle j+1, m, n|X^3|j, m, n\rangle &= q^{m-j-\frac{1}{2}}\sqrt{1+q^2}\sqrt{[j-m+1][j+m+1]} \\ &\quad \times\langle j+1, n\|X^-\|j, n\rangle \\ \langle j-1, m, n|X^3|j, m, n\rangle &= -q^{m+j+\frac{1}{2}}\sqrt{1+q^2}\sqrt{[j-m][j+m]}\langle j-1, n\|X^-\|j, n\rangle. \end{aligned} \tag{4.6}$$

The matrix elements on the rhs are the reduced ones. Using conjugation properties (2.10) we have

$$\langle j+1, n\|X^-\|j, n\rangle = -q^{2j+2}\overline{\langle j, n\|X^-\|j+1, n\rangle}. \tag{4.7}$$

Therefore only one reduced matrix element has to be determined, which is easily obtained from the first equation of (2.3) and (4.2). We fix the phase by setting

$$\langle j+1, n\|X^-\|j, n\rangle = \frac{l_0 q^{j+2n}}{\sqrt{[2][2j+1][2j+3]}}. \tag{4.8}$$

Incidentally, the first equation of (2.3) also tells us that $\langle j, n\|X^-\|j, n\rangle$ must vanish.

Now we come to the matrix elements of P^A. Based on equations (4.6) and (4.7) they are calculable by relying on the matrix elements of the values $X \circ P$ and its conjugate $P \circ X$. The Heisenberg relation (2.20) with the help of the R matrix (2.9) yields after contraction

$$P \circ X - q^6 X \circ P = -\frac{\mathrm{i}}{2}\lambda^{-\frac{1}{2}}(1+q^{-6})(q^2+1+q^{-2})W. \tag{4.9}$$

Together with its conjugation equation (4.9) gives

$$X \circ P = -\frac{\mathrm{i}}{2}\frac{(\Lambda^{\frac{1}{2}} - \Lambda^{-\frac{1}{2}})W}{q^2(q^2-1)}$$
$$P \circ X = \frac{\mathrm{i}}{2}\frac{(q^{-6}\Lambda^{-\frac{1}{2}} - q^6\Lambda^{\frac{1}{2}})W}{q^2(q^2-1)}. \tag{4.10}$$

Therefore $X \circ P$ has matrix elements only between neighbouring n. We consider now

$$\begin{aligned}\langle j, m, n|X \circ P|j, m, n+1\rangle = -q^2\{&[2j+3][2j+2]\\ &\times\langle j, n\|X^-\|j+1, n\rangle\langle j+1, n\|P^-\|j, n+1\rangle + [2j][2j-1]\\ &\times\langle j, n\|X^-\|j-1, n\rangle\langle j-1, n\|P^-\|j, n+1\rangle\} = -\frac{\mathrm{i}}{2}\frac{W_j}{q^5(q^2-1)}\end{aligned} \tag{4.11}$$

where the reduced matrix elements of P^A are defined analogously to equations (4.6), including the fact that they are no longer diagonal in n. Now it is straightforward to take

$$\begin{aligned}\langle j, m, n+1|X \circ P|j, m, n\rangle = -q^2\{&[2j+3][2j+2]\\ &\times\langle j, n+1\|X^-\|j+1, n+1\rangle\langle j+1, n+1\|P^-\|j, n\rangle + [2j][2j-1]\\ &\times\langle j, n+1\|X^-\|j-1, n+1\rangle\langle j-1, n+1\|P^-\|j, n\rangle\} = \frac{\mathrm{i}}{2}\frac{W_j q}{q^2-1}.\end{aligned} \tag{4.12}$$

We put in equations (4.6) and (4.7) and the conjugation relations

$$\begin{aligned}\langle j+1, n\|P^-\|j, n+1\rangle &= -q^{2j+2}\langle j, n+1\|P^-\|j, n\rangle\\ \langle j+1, n+1\|P^-\|j, n\rangle &= -q^{2j+2}\langle j, n\|P^-\|j+1, n+1\rangle.\end{aligned} \tag{4.13}$$

The system (4.11) and (4.12) can be rewritten as two recursion relations in j for the two unknowns, the reduced matrix elements of P. An easy way to solve it is to start with $j = 0$, read off the general formula and prove it by insertion. For clarity, we present all non-vanishing reduced matrix elements

$$\begin{aligned}\langle j+1, n\|P^-\|j, n+1\rangle &= -\mathrm{i}q^{-j-6-2n}Z^{-1}\\ \langle j, n+1\|P^-\|j+1, n\rangle &= -\mathrm{i}q^{-3j-8-2n}Z^{-1}\\ \langle j+1, n+1\|P^-\|j, n\rangle &= \mathrm{i}q^{3j-2-2n}Z^{-1}\\ \langle j, n\|P^-\|j+1, n+1\rangle &= \mathrm{i}q^{j-4-2n}Z^{-1}\end{aligned} \tag{4.14}$$

where the common denominator is

$$Z = 2l_0\lambda\sqrt{[2][2j+1][2j+3]}.$$

Neither equation (4.8) nor (4.14) contain any divergencies because of the condition $j_{\max} \leqslant \frac{r}{2} - 1$. If $j+1$ exceeds $j_{\max}$ the matrix element simply vanishes as it does for $j - 1 = -1$.

Our next aim is to calculate the eigenvalues of $P^2 \equiv P \circ P$. We shall follow the lines of section 3 and start with the definition of a unitary operator

$$U = q^3\Lambda^{\frac{1}{2}}. \tag{4.15}$$

Going back to equation (4.3) we have

$$U|n\rangle = |n-1\rangle \tag{4.16}$$

where we have omitted all quantum numbers which are unchanged. After

$$U|0\rangle = \mathrm{e}^{\mathrm{i}\alpha}|r-1\rangle \tag{4.17}$$

we introduce

$$\begin{aligned}U' &= U\mathrm{e}^{-\frac{\mathrm{i}\alpha}{r}}\\ U'|n\rangle' &= |n-1\rangle'.\end{aligned} \tag{4.18}$$

The eigenstates of the operator U' are given by

$$|\phi_k\rangle = \sum_{n=0}^{r-1} q^{2nk}|n\rangle' \qquad U'|\phi_k\rangle = q^{2k}|\phi_k\rangle. \tag{4.19}$$

Note that the eigenstates for even k can be produced by the operator $X^2/{l_0}^2$ acting $k/2$ times on $|\phi_0\rangle$. From the algebra (2.18) follows

$$P \circ P|\phi_k\rangle = \tilde{d}_k|\phi_{k-2}\rangle \tag{4.20}$$

where we shall calculate $\tilde{d}_k$ below. For the P-eigenstates we use the ansatz

$$P \circ P|p_n\rangle = p_n^2|p_n\rangle \qquad |p_n\rangle = \sum_{k=0}^{r-1} a_k|\phi_k\rangle. \tag{4.21}$$

Equation (4.20) yields the recursion relation

$$a_{k+2} = \frac{p_m^2}{\tilde{d}_{k+2}} a_k. \tag{4.22}$$

Now it is necessary to distinguish between even and odd r. In the first case we obtain two different solutions putting $a_0 = 1, a_1 = 0$ and vice versa. They contain either even or odd numbers of k in the sum (4.21). Consistency gives for the eigenvalues

$$(p_+^2)^{\frac{r}{2}} = \prod_{k=0}^{\frac{r}{2}-1} \tilde{d}_{2k} \qquad (p_-^2)^{\frac{r}{2}} = \prod_{k=0}^{\frac{r}{2}-1} \tilde{d}_{2k+1}. \tag{4.23}$$

For odd r the sum (4.21) contains all numbers and hence

$$(p^2)^r = \prod_{k=0}^{r-1} \tilde{d}_k. \tag{4.24}$$

The coefficients $\tilde{d}_k$ are calculated via the matrix elements of P^2 between the $|j, n\rangle$ states. We have the same structure as in the first parts of equations (4.11) and (4.12), e.g.

$$\begin{aligned}\langle j, n+2|P^2|j, n\rangle = -q^2\{&[2j+3][2j+2] \\ &\times\langle j, n+2\|P^-\|j+1, n+1\rangle\langle j+1, n+1\|P^-\|j, n\rangle \\ &+[2j][2j-1]\langle j, n+2\|P^-\|j-1, n+1\rangle\langle j-1, n+1\|P^-\|j, n\rangle\}.\end{aligned} \tag{4.25}$$

With the results of equation (4.14) we obtain

$$\langle j, n+2|P^2|j, n\rangle = -\frac{q^{-4n-10}}{4l_0^2\lambda^2} \tag{4.26}$$

and in the same way

$$\langle j, n-2|P^2|j, n\rangle = -\frac{q^{-4n-2}}{4l_0^2\lambda^2}. \tag{4.27}$$

A little more lengthy is the calculation of the diagonal element due to the doubling of terms connected with intermediate states having quantum numbers $n \pm 1$.

$$\langle j, n|P^2|j, n\rangle = \frac{q^{-4n-6}}{4l_0^2\lambda^2}\{4j+2\}. \tag{4.28}$$

As soon as the quantum numbers of the rhs ket vector are fixed there are no further non-vanishing matrix elements. Now we consider

$$\begin{aligned} P^2|n\rangle' &= |n\rangle'\langle n|'P^2|n\rangle' + |n+2\rangle'\langle n+2|'P^2|n\rangle' + |n-2\rangle'\langle n-2|'P^2|n\rangle' \\ &= |n\rangle'\langle n|P^2|n\rangle + |n+2\rangle'\mathrm{e}^{-\frac{2i\alpha}{r}}\langle n+2|P^2|n\rangle + |n-2\rangle'\mathrm{e}^{\frac{2i\alpha}{r}}\langle n-2|P^2|n\rangle. \end{aligned} \tag{4.29}$$

From equation (4.20) follows

$$\begin{aligned} P^2|\phi_k\rangle &= \sum_{n=0}^{r-1} q^{2nk}|n\rangle'(q^{-4k}\mathrm{e}^{-\frac{2i\alpha}{r}}\langle n+2|P^2|n\rangle + q^{4k}\mathrm{e}^{\frac{2i\alpha}{r}}\langle n-2|P^2|n\rangle + \langle n|P^2|n\rangle) \\ &= \tilde{d}_k \sum_{n=0}^{r-1} q^{2nk-4n}|n\rangle'. \end{aligned} \tag{4.30}$$

Substituting equations (4.26)–(4.28) we can read off

$$\begin{aligned} \tilde{d}_k &= \frac{q^{-6}}{4l_0^2\lambda^2}(\{4j+2\} - q^{-4k-4}\mathrm{e}^{-\frac{2i\alpha}{r}} - q^{4k+4}\mathrm{e}^{\frac{2i\alpha}{r}}) \\ &= -\frac{q^{-6}}{4l_0^2}[2k+2j+3]_\alpha[2k-2j+1]_\alpha \end{aligned} \tag{4.31}$$

where we have introduced the abbreviation

$$[a]_\alpha = \frac{q^a\mathrm{e}^{\frac{i\alpha}{r}} - q^{-a}\mathrm{e}^{-\frac{i\alpha}{r}}}{\lambda}. \tag{4.32}$$

Finally we have for even r

$$(p_\pm^2)^{\frac{r}{2}} = \frac{-\mathrm{i}^r}{2^r\lambda^r l_0^r}(-1)^{\frac{r}{2}} f(q,\alpha) f(q,\alpha-\pi r) \tag{4.33}$$

and for odd r

$$(p^2)^r = \frac{-\mathrm{i}^{2r}}{2^{2r}\lambda^{2r} l_0^{2r}} f^2(q,\alpha) f^2(q,\alpha-\pi r). \tag{4.34}$$

While $\tilde{d}_k$ depends on j, p^2, of course, does not. It is remarkable that equations (4.33) and (4.34) greatly resemble equation (3.20) derived for the one-dimensional model.

For even r any eigenvalue is degenerated twice, disregarding the obvious degeneration with respect to j and m. All eigenvectors (4.21) are orthogonal and normalizable. (Note that this is not true for the $|\phi_k\rangle$ states.) The eigenvalues of P^2 are in both cases given by even powers of q multiplied by the roots of real functions on q. The main difference from real q is the finiteness in dimension for the eigenvector space.

It would be interesting to know more about the function $f(q,\alpha)$; in particular, it should play a role in a generalized Fourier transformation. We hope to return to this question in our further work. Our experience for finite r seems to lead to the conjecture that the roots in equations (4.33) and (4.34) can be easily extracted if one excludes all polynomials which vanish after being multiplied with non-zero combinations of powers of q.

At the end of this section we shall return to the question of how the structure we have found is related to former attempts of combining non-commutative geometry with string theory via a matrix realization [14].

Our operators X^i can be viewed as matrices acting on vectors with dimension $\frac{r^3}{4}$ (for even r) as long as l_0 is kept fixed. It is natural to ask whether they can be considered as analogues of the X_i fields ($0 \leqslant i \leqslant 9$) in the IKKT model [13]. The role played there by SO(10, C) is played here by SO(3).

Nevertheless there are substantial differences between the two sets of operators. Even though there is an analogue of their unitary gauge fields U_i, namely the unitary operator $q^3\Lambda^{\frac{1}{2}}$ (this is the only one), that operator plays a different role. Its 'gauge transformation' induces a multiplicative rescaling while in paper [14] the gauge transformation adds a constant (proportional to the compactification radius). It is this fact which requires infinite matrices in the IKKT model in order to have an infinite trace, while direct calculation yields $\mathrm{Tr}\, X^i = 0$ in our case. This discrepancy becomes less important when remembering that for the full trace one has to integrate over l_0, which produces a divergent result.

We propose to solve the remaining problems by taking into account the fact that in the IKKT model the undeformed group $\mathrm{SO}(10, C)$ was used while we started with the q-deformed $\mathrm{SO}_q(3)$.

Concluding this remark we state that we have found a self-consistent structure which is close to becoming an analogue of an IKKT-like matrix model on a non-commutative torus. This problem is being worked on now.

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References

[1] Carow-Watamura U, Schlieker M and Watamura S 1991 *Z. Phys.* C **49** 439
[2] Carow-Watamura U, Schlieker M, Watamura S and Weich W 1991 *Commun. Math. Phys.* **142** 605
[3] Lorek A, Weich W and Wess J 1997 *Z. Phys.* C **76** 375
[4] Connes A 1994 *Noncommutative Geometry* (New York: Academic)
[5] Kirillov A N and Reshetikhin N Yu 1991 *New Developments in the Theory of Knots* vol 202, ed T Kohno (Singapore: World Scientific)
[6] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[7] Jüttner G and Karowski M 1994 *Nucl. Phys.* B **430** 615
[8] Faddeev L D, Reshetikhin N Yu and Takhatajan L A 1987 *Alg. Ana.* **1** 178
[9] Hebecker A, Schreckenberg S, Schwenk J and Wess J 1994 *Z. Phys.* C **64** 355
[10] Schwenk J 1994 *Preprint* MPI-PhT/94-36, hep-th/9406168
[11] Cerchiai B L and Wess J 1998 *Preprint* MPI-PhT/98-09
[12] Pasquier V 1988 *Commun. Math. Phys.* **118** 178
[13] Ishibashi N, Kawai H, Kitazawa Y and Tsuchiya A 1997 *Nucl. Phys.* B **498** 467
[14] Connes A, Douglas M and Schwarz A 1997 *Preprint* IHES/P/97/82, hep-th/9711162